Range-Based Algorithm for Max-CSP

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Abstract. A Max-CSP consists of searching for a solution which minimizes the number of violated constraints. The best existing solving algorithm is PFC-MRDAC. It is based on the computation of a lower bound of the number of violations. To compute this lower bound it is required to evaluate the violations with respect to each value of each domain. Unfortunately, some applications imply thousands of variables with huge domains. In scheduling, it arises that numerous activities have to be scheduled over several months with a unit of time of a few minutes. In this situation using PFC-MRDAC requires a large amount of memory which can prevent from using it. In this paper, we propose an algorithm called the Range-based Max-CSP Algorithm (RMA), based on the exploitation of bound-based filtering algorithms of constraints. This technique does not require to study each value of each domain: its complexity depends only on the number of variables and the number of constraints. No assumption is made on the constraints except that their filtering algorithms are related to the bounds of the involved variables, the general case for scheduling constraints. Then, when the only information available for a variable x w.r.t. a constraint C are the new bounds of $D(x)$ obtained by applying the filtering algorithm of C, the lower bounds of violations provided by PFC-MRDAC and RMA are identical.

1 Introduction

A problem is over-constrained when no assignment of values to variables satisfies all constraints. The simplest theoretical framework for over-constrained problems is the Maximal Constraint Satisfaction Problem (Max-CSP). A solution of a Max-CSP is a total assignment of values to variables that minimizes the number of constraint violations.

The best solving algorithm for Max-CSP, PFC-MRDAC [1], has been extended to the non binary case [4, 5, 3]. Moreover, in [4, 5] it has been shown that a Max-CSP can be encoded through a global constraint. When we define a Max-CSP as a global constraint, then we use the PFC-MRDAC algorithm as the filtering algorithm of the constraint. This formulation is suited to real-life problems, because generally some parts correspond to Max-CSPs although the whole problem cannot be expressed as a Max-CSP. Moreover any search algorithm can be used. Therefore we selected it to describe the algorithms presented in this paper.

For sake of clarity, we consider first that a Max-CSP is solved by a Branch and Bound based search algorithm: successive assignments of values to variables are performed through a depth-first traversal of the search tree, where internal nodes represent incomplete assignments and leaf nodes stand for complete ones. The number of violations to minimize is expressed by an objective variable obj. For any given node, $UB = max(D(obj)) + 1$ corresponds to the number of violations of the best solution found so far. $min(D(obj))$ is the number of violated constraints detected by the solver at the current step of the search. In fact, $min(D(obj))$ can be simply defined as the number of constraints C such that values have been assigned to all variables in the set $var(C)$ of the variables involved in C, and C is violated. This number is generally called the *distance*. If $distance > max(D(obj))$ then the current best solution cannot be improved below the current node. Thus it is not necessary to traverse the sub-tree rooted at the current node.

PFC-MRDAC improves that condition by computing a lower bound LB of the number of violations, equal to *distance* plus an under-estimation of the number of violations entailed by constraints involving some variables which have not yet been instantiated. The new condition of continuation of the search is $LB \leq max(D(obj))$.

When filtering, PFC-MRDAC combines generally LB with a lower bound local to each value, in order to remove this value if it cannot belong to a solution.

All these lower bounds are based on direct violations of constraints by values: they require to maintain for each value a in the domain of a variable x the number of constraints C such that a is not consistent with C. This principle is applicable to a wide number of problems, providing that domain sizes are not too big.

Unfortunately, some applications involve a big number of variables with huge domain sizes.

For instance, consider the problem of scheduling the construction of offices. Some activities, as painting or installing windows, have a short duration and require essentially human resources. They can be performed over a large period. They concern each office of each floor in each building, and they are linked to other activities by precedence rules. Given that the manager has to deal with a large number of workers who express preference constraints with respect to their week schedule, it is mandatory to have a time unit of at most one quarter of hour. If the total duration of the project is one year then for each activity we have 35040 possible start dates. If there exists thousands of activities then it is not realistic to maintain for each possible date of each activity a counter of inconsistencies. Moreover, since scheduling constraints generally propagate on bounds, redundant computations will be made when testing the consistency of each value of a domain with a constraint. Therefore, PFC-MRDAC is not really suited to this case.

In this paper, we propose a new algorithm called the Range-based Max-CSP Algorithm (RMA), which computes a lower bound by exploiting filtering algorithms of constraints. The principle is similar to PFC-MRDAC except concerning the computation of inconsistencies involved by each variable. Instead of evaluating each value of a domain $D(x)$, the RMA stores two entities per constraint C in which x is involved: the new minimum and maximum of $D(x)$ obtained when applying the filtering algorithm of C. For instance, for 100 variables with domains of 35040 values and 100 constraints, the RMA requires to maintain at most 200 minima and maxima per variable (corresponding to a worst-case situation where all constraints imply all variables) instead of 35040 counters in PFC-MRDAC. That is, instead of having a total of 3.5 millions of counters with PFC-MRDAC we have 20000 entities with the RMA.

We show that in the case of constraints which propagate only bounds of domains, the same LB is obtained by using PFC-MRDAC and the RMA.

We compare the complexity of a non incremental implementation with PFC-RDAC:[1] when domain sizes are big the amortized complexity of the RMA is better because it does not depend on the number of values in the domains.

We provide a filtering algorithm based on the same principle, which propagates on bounds. We discuss the incremental version of the RMA and about a filtering algorithm which performs holes in domains. We point out a promising generalization of our algorithm: for general problems, it is possible to take into account ranges of successive values involving the same number of violations to compute the lower bounds.

2 Preliminaries

CSP A *constraint network* $\mathcal{N}$ is defined as a set of n *variables* $\mathcal{X} = \{x_1, \ldots, x_n\}$, a set of *domains* $\mathcal{D} = \{D(x_1), \ldots, D(x_n)\}$ where $D(x_i)$ is the finite set of possible *values* for variable x_i, and a set $\mathcal{C}$ of *constraints* between variables. A *constraint* C on the ordered set of variables $var(C) = (x_{i_1}, \ldots, x_{i_r})$ (also denoted by $C(x_{i_1}, \ldots, x_{i_r})$) is defined by the subset $rel(C)$ of the Cartesian product $D(x_{i_1}) \times \cdots \times D(x_{i_r})$ that specifies the *allowed* combinations of values for the variables $x_{i_1}, \ldots, x_{i_r}$. $D(var(C)) = \cup_{x \in var(C)} D(x)$. An element of $D(x_{i_1}) \times \cdots \times D(x_{i_r})$ is called a *tuple on* $var(C)$. $|var(C)|$ is the *arity* of C. C is *binary* iff $|var(C)| = 2$. A value a for a variable x is denoted by (x, a). A tuple τ on $var(C)$ is *valid* if $\forall (x, a) \in \tau, a \in D(x)$. C is *consistent* iff there exists a tuple τ of $rel(C)$ which is valid. A value $a \in D(x)$ is *consistent with* C iff $x \notin var(C)$ or there exists a valid tuple τ of $rel(C)$ in which a is the value assigned to x. Given $Y \subseteq \mathcal{X}$, an *instantiation* I of Y is an assignment of values to variables Y such that $\forall x \in Y$, the value a assigned to x belongs to $D(x)$. Given $Y \subseteq \mathcal{X}$ and $C \in \mathcal{C}$ such that $var(C) \subseteq Y$, an instantiation I of Y *satisfies* a constraint C iff the projection of I on $var(C)$ belongs to $rel(C)$. If I does not satisfy C, then I *violates* C. The *Constraint Satisfaction Problem* (CSP) consists of finding an instantiation I of X such that $\forall C \in \mathcal{C}$, I satisfies C.

Over-Constrained problem When a CSP has no solution, we say that the problem is *over-constrained.* $\mathcal{C}_h \subseteq \mathcal{C}$ is the set of *hard* constraints, that is, the

[1] PFC-RDAC [2] is the non incremental version of PFC-MRDAC.

constraints that must necessarily be satisfied. $\mathcal{C}_s = \mathcal{C} \setminus \mathcal{C}_h$ is the set of *soft* constraints. Let I be an instantiation of X. If I is a solution of an over-constrained problem then $\forall C \in \mathcal{C}_h$, I satisfies C.

Max-CSP The Max-CSP is the problem where $\mathcal{C}_h = \emptyset$ and the goal is to find an assignment of values to variables that minimizes the number of violations in $\mathcal{C} = \mathcal{C}_s$.

Max-CSP as a global constraint Let $\mathcal{N} = (\mathcal{X}, \mathcal{D}, \mathcal{C})$ be a constraint network. Constraints in $\mathcal{C}$ can be encapsulated into a single constraint [4, 5], called the Satisfiability Sum Constraint (*ssc*):

Definition 1 *Let $\mathcal{C} = \{C_i, i \in \{1, \dots, m\}\}$ be a set of constraints, and $S[\mathcal{C}] = \{s_i, i \in \{1, \dots, m\}\}$ be a set of variables and obj be a variable, such that a one-to-one mapping is defined between $\mathcal{C}$ and $S[\mathcal{C}]$. A* **Satisfiability Sum Constraint** *is the constraint $ssc(\mathcal{C}, S[\mathcal{C}], obj)$ defined by:*

$$[obj = \sum_{i=1}^{m} s_i] \wedge \bigwedge_{i=1}^{m} [(C_i \wedge (s_i = 0)) \vee (\neg C_i \wedge (s_i = 1))]$$

Notation 1 *Given a $ssc(\mathcal{C}, S[\mathcal{C}], obj)$, a variable x, $a \in D(x)$ and $\mathcal{K} \subseteq \mathcal{C}$:*
- *$max(D(obj))$ is the highest value of current domain of obj;*
- *$min(D(obj))$ is the lowest value of current domain of obj;*
- *$S[\mathcal{K}]$ is the subset of $S[\mathcal{C}]$ equals to the projection of variables $S[\mathcal{C}]$ on $\mathcal{K}$;*
- *$var(\mathcal{C})$ is the union of $var(C_i), C_i \in \mathcal{C}$.*

The variables $S[\mathcal{C}]$ are used to express which constraints of $\mathcal{C}$ must be violated or satisfied: value 0 assigned to $s \in S[\mathcal{C}]$ expresses that its corresponding constraint C is satisfied, whereas 1 expresses that C is violated. Throughout this formulation, a solution of a Max-CSP is an assignment that satisfies the *ssc* with the minimal possible value of *obj*. Comparing a lower bound of the objective to $max(D(obj))$ leads to a necessary consistency condition of the *ssc*. Domain reduction algorithms for the Max-CSP correspond to specific filtering algorithms of the *ssc*.

Definition 1. *Given a $ssc(\mathcal{C}, S[\mathcal{C}], obj)$ and a variable x:*
- *$minObj(\mathcal{C}, S[\mathcal{C}])$ is the minimum value of obj consistent with $ssc(\mathcal{C}, S[\mathcal{C}], obj)$;*
- *$minObj((x, a), \mathcal{C}, S[\mathcal{C}])$ is equal to $minObj(\mathcal{C}, S[\mathcal{C}])$ when $x = a$;*

3 Generalized version of PFC-MRDAC

PFC-MRDAC [1] is one of the best algorithm for solving binary Max-CSPs. This section is a summary of the generalization of this algorithm to non-binary constraints [4, 5, 3]. Basically, PFC-MRDAC is based on counters of violations involved by each value of each domain:

Definition 2 *Let x be a variable, a be a value of $D(x)$, $\mathcal{C}$ be a set of constraints, $\#inc((x, a), \mathcal{C}) = |\{C \in \mathcal{C}$ s.t. (x, a) is not consistent with $C\}|$.*

In a solver, this information can be obtained by applying independently the specific filtering algorithms of the constraints in which x is involved, while the domain of x is reduced to the value a we study.

3.1 Necessary condition of consistency

From the definition of $minObj(\mathcal{C}, S[\mathcal{C}])$ we have:

Property 1 *If $minObj(\mathcal{C}, S[\mathcal{C}]) > max(D(obj))$ then $ssc(\mathcal{C}, S[\mathcal{C}], obj)$ is not consistent.*

A lower bound of $minObj(\mathcal{C}, S[\mathcal{C}])$ provides a necessary condition of consistency of a *ssc*. A possible way for computing it is to perform a sum of independent lower bounds of violations, one per variable. For each variable a lower bound can be defined by:

Definition 3 *Given a variable x and a constraint set $\mathcal{K}$, $\#inc(x, \mathcal{K}) = min_{a \in D(x)}$ $(\#inc((x, a), \mathcal{K}))$.*

The sum of minima $\#inc(x, \mathcal{K})$ over all the variables in $\mathcal{K} = \mathcal{C}$ cannot lead to a lower bound of the total number of violations. Indeed, some constraints can be taken into account more than once. In this case, the lower bound can be overestimated, and an inconsistency could be detected while the *ssc* is consistent.

In the binary case, the constraint graph[2] is used in order to guarantee this independence [1]. Each edge is oriented and for each variable x only the constraints out-going x are taken into account.

This idea was be generalized to the non binary case, by associating with each constraint C one and only one variable x involved in the constraint [4, 5]: C is then taken into account only for computing the $\#inc$ counter of x. Therefore, the constraints are partitioned according to the variables they are associated with:

Definition 4 *Given a set of constraints $\mathcal{C}$, a* **var-partition** *of $\mathcal{C}$ is a partition $\mathcal{P}(\mathcal{C}) = \{P(x_1), ..., P(x_k)\}$ of $\mathcal{C}$ in $|var(\mathcal{C})|$ sets such that $\forall P(x_i) \in \mathcal{P}(\mathcal{C}) : \forall C \in P(x_i), x_i \in X(C)$.*

Given a var-partition $\mathcal{P}(\mathcal{C})$, the sum of all $\#inc(x_i, P(x_i))$ is a lower bound of the total number of violations, because all sets belonging to $\mathcal{P}(\mathcal{C})$ are disjoint; thus we obtain the following lower bound:

$$LB_{\mathcal{P}(\mathcal{C})} = \sum_{x_i \in var(\mathcal{C})} \#inc(x_i, P(x_i))$$

Property 2 $\forall \mathcal{P}(\mathcal{C}) = \{P(x_1), ..., P(x_k)\}$, $LB_{\mathcal{P}(\mathcal{C})} \leq minObj(\mathcal{C}, S[\mathcal{C}])$.

The necessary condition of consistency of a *ssc* is deduced from this Property:

Corollary 1 $\forall \mathcal{P}(\mathcal{C}) = \{P(x_1), ..., P(x_k)\}$, *If $LB_{\mathcal{P}(\mathcal{C})} > max(D(obj))$ then $ssc(\mathcal{C}, S[\mathcal{C}], obj)$ is not consistent.*

[2] the graph where vertices are variables and edges are binary constraints between pairs of variables.

3.2 Filtering algorithm

PFC-MRDAC [1] and its extension to the non binary case [4, 5, 3] include a look-ahead procedure used to reduce domains of variables that have not yet an assigned value. From definition of $minObj((x,a),\mathcal{C},S[\mathcal{C}])$ we have:

Theorem 1 *$\forall x \in var(\mathcal{C}), \forall a \in D(x)$: if $minObj((x,a),\mathcal{C},S[\mathcal{C}]) > max(D(obj))$ then (x,a) is not consistent with ssc $(\mathcal{C}, S[\mathcal{C}], obj)$.*

Any lower bound of $minObj((x,a),\mathcal{C},S[\mathcal{C}])$ can be used to check the consistency of (x,a). An obvious lower bound is $\#inc((x,a),\mathcal{C})$:

Property 3 $\#inc((x,a),\mathcal{C}) \leq minObj((x,a),\mathcal{C},S[\mathcal{C}])$

From this Property and Theorem 1, we obtain a first filtering algorithm. It can be improved by including the lower bound of Property 2. This idea was introduced by Larrosa et al. [1], for binary constraint networks.

It can be applied in the general case [4, 5, 3]. In order to do so, we suggest to split $\mathcal{C}$ into two disjoint sets $P(x)$ and $\mathcal{C} - P(x)$, where $P(x)$ is the subset of constraints associated with x in a var-partition $\mathcal{P}(\mathcal{C})$ of $\mathcal{C}$. Consider the following corollary of Theorem 1:

Corollary 2 *Let $\mathcal{P}(\mathcal{C})$ be a var-partition of $\mathcal{C}$, x a variable and $a \in D(x)$, if $minObj((x,a),P(x),S[P(x)]) + minObj((x,a),\mathcal{C} - P(x),S[\mathcal{C} - P(x)]) > max(D(obj))$ then (x,a) is not consistent with $ssc(\mathcal{C},S[\mathcal{C}],obj)$.*

Proof: e.g., [5].

Note that $minObj(\mathcal{C} - P(x),S[P(x)]) \leq minObj((x,a),\mathcal{C} - P(x),S[P(x)])$. From this remark and Properties 2 and 3 the following theorem can be deduced:

Theorem 2 *$\forall \mathcal{P}(\mathcal{C})$ a var-partition of $\mathcal{C}, \forall x \in var(\mathcal{C}), \forall a \in D(x)$, if $\#inc((x,a), P(x)) + LB_{\mathcal{P}(\mathcal{C}-P(x))} > max(D(obj))$ then a can be removed from $D(x)$.*

4 The Range-based Max-CSP Algorithm

4.1 Principle

PFC-MRDAC algorithm requires to maintain one counter $\#inc$ $((x,a),\ P(x))$ for each value a of each domain $D(x)$. The computation of these counters is sufficient to efficiently compute the consistency of a satisfiability sum constraint (see Corollary 1) and to apply the filtering algorithm associated with it (see Theorem 2). Therefore, the main issue of the PFC-MRDAC algorithm is the computation of these counters, and if this computation is accelerated then the algorithm is improved.

We propose a new algorithm called the Range-based Max-CSP Algorithm (RMA) that does not require one counter per value although the principle is very similar to the non binary PFC-MRDAC: for each variable x we consider a set $P(x)$ of a var-partition of constraints $\mathcal{P}(\mathcal{C})$, in order to compute a lower bound $\#inc(x, P(x))$ of the violations involved by x.

The difference is the way we compute $\#inc(x, P(x))$. In the RMA, there is no need to maintain one counter of violation $\#inc((x, a), P(x))$ for each value a in $D(x)$ as it is the case in PFC-MRDAC (see Definition 3).

The idea exploited in our algorithm is based on the following definition:

Definition 5 *Let $I \subseteq D(x)$ be a range of consecutive values. If $\forall a \in I$, $\forall b \in I$, $\#inc((x, a), P(x)) = \#inc((x, b), P(x))$ then I is homogeneously inconsistent. The number of constraints in $P(x)$ violated if $D(x) = I$ is $\#inc(I, P(x)) = \#inc((x, a), P(x)), a \in I$.*

If any value in a range I violates the same number of constraints of $P(x)$ then we can consider globally the range to evaluate the number of violations, instead of studying the values one by one.

It is possible to take into account only a set of homogeneously inconsistent ranges to compute $\#inc(x, P(x))$, provided that each value of $D(x)$ belongs to one range. More formally, we have:

Definition 6 *A set $\mathcal{I}(P(x)) = \{I_1, ..., I_m\}$ of homogeneously inconsistent ranges such that $\cup_{k=1}^{m} I_k = D(x)$ is called a set of homogeneously inconsistent ranges which covers $D(x)$.*

Property 4 *Let $\mathcal{I}(P(x))$ be a set of homogeneously inconsistent ranges which covers $D(x)$. Then we have: $\#inc(x, P(x)) = min_{I \in \mathcal{I}(P(x))} \#inc(I, P(x))$.*

Proof: from Definition 5

Thus, if we are able to identify such a set $\mathcal{I}(P(x))$ with $|\mathcal{I}(P(x))| < |D(x)|$ then it is possible to improve PFC-MRDAC in two ways:

1. the number of counters required to compute $\#inc(x, P(x))$ is smaller,
2. $\#inc(x, P(x))$ can be computed faster.

The first point is quite important in practice. Indeed, in some problems the number of variables and the size of the domains make methods requiring for each value of each variable some additional data, as $\#inc((x, a), P(x))$, unusable in practice. For instance, this is the case of almost all real world scheduling applications. Moreover, in such problems, the filtering algorithms associated with constraints are range-based filtering algorithms: they reduce only the bounds of the domain. They do not create "holes" in domains.

Such a range-based approach is interesting if some issues can be efficiently solved:

- the number of ranges is small,
- for each range I, $\#inc(I, P(x))$ can be efficiently computed.

In the remaining of this section, we will consider that only range-based filtering algorithms are used. Under this condition, we will prove that:

- the size of $\mathcal{I}(P(x))$ is at most $2 * |P(x)| + 1$,
- the set of all $\#inc(I, P(x))$ can be computed in $O(|P(x)|)$ provided that we have computed $\mathcal{I}(P(x))$.

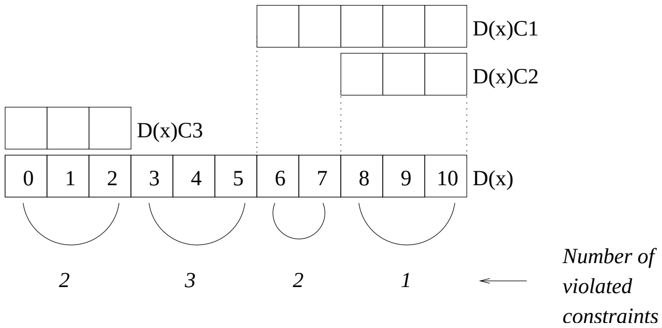
4.2 Computation of $\#inc$ counters

The principle is to consider the bounds obtained by applying independently each filtering algorithm of constraints in $P(x)$ on $D(x)$:

Notation 2 *Let $\mathcal{P}(\mathcal{C})$ be a var-partition, x a variable, $P(x) \in \mathcal{P}(\mathcal{C})$ and $C \in P(x)$: $D(x)_C$ is the domain obtained by applying the filtering algorithm of C on $D(x)$.*

For each constraint C we consider the minimum $min(D(x)_C)$ and the maximum $max(D(x)_C)$ of $D(x)_C$. By ordering all these minima and maxima from the lower to the greater, it is possible to divide $D(x)$ in different ranges. Each of them corresponds to a certain number of violations. The following figure illustrates an example where $P(x)$ is the set of constraints $\{C_1, C_2, C_3\}$, which involve four variables x, y, z, t such that $D(x) = D(y) = D(z) = D(t) = [0, 10]$:

- $C_1 = [x - y > 5]$ that leads to $min(D(x)_{C_1}) = 6$, $max(D(x)_{C_1}) = 10$,
- $C_2 = [x - z > 7]$ that leads to $min(D(x)_{C_2}) = 8$, $max(D(x)_{C_2}) = 10$,
- $C_3 = [t - x > 7]$ that leads to $min(D(x)_{C_3}) = 0$, $max(D(x)_{C_3}) = 2$.



In this example the homogeneously inconsistent ranges are $[0, 3)$, $[3, 6)$, $[6, 8)$, $[8, 11)$ and $\#inc(x, P(x)) = \#inc(x, P(x)) = min_{I \in \mathcal{I}(P(x))} \#inc(I, P(x)) = min(\#inc([0, 3), P(x)) = 2, \#inc([3, 6), P(x)) = 3, \#inc([6, 8), P(x)) = 2, \#inc([8, 11), P(x)) = 1) = 1$.

Definition 7 *Let $B(P(x))$ be the set containing the bounds of $D(x)$ plus the bounds of $D(x)_C$ for all constraints C in $P(x)$, sorted in ascending order. We denote by $\mathcal{I}(B(P(x)))$ the set of ranges such that each range is defined by a pair of two consecutive elements of $B(P(x))$. The k^{th} range of $\mathcal{I}(B(P(x)))$ is denoted by $I_k = [p, q)$, where p is the minimum of I_k and $q - 1$ is the maximum.*

Property 5 *The following properties hold:*

1. *the maximal possible number of ranges in* $\mathcal{I}(B(P(x)))$ *is* $2*|P(x)|+1$
2. $\forall a \in D(x)\ \exists I \in \mathcal{I}(B(P(x)))$ *such that* $a \in I$
3. $\forall I \in \mathcal{I}(B(P(x))) \forall C \in P(x) : I \subseteq D(x)_C$ *or* $I \cap D(x)_C = \emptyset$
4. $\forall I \in \mathcal{I}(B(P(x)))$*,* I *is homogeneously inconsistent.*
5. $\mathcal{I}(B(P(x)))$ *is a set of homogeneously inconsistent ranges which covers* $D(x)$*.*

Proof:

1. $B(P(x))$ contains at most $2*|P(x)|+2$ values, thus the maximum number of ranges is $2*|P(x)|+1$.
2. By construction of $\mathcal{I}(B(P(x)))$
3. Suppose that $I \cap D(x)_C = J$ with $J \neq \emptyset$ and $J \neq I$. Then, one bound of J is equal to a bound of $D(x)_C$ and is not equal to a bound of I. So, I contains a bound of $D(x)_C$ which is not one of its bounds. This is not possible by construction of $\mathcal{I}(B(P(x)))$.
4. From 3 $\forall C \in P(x)$ and $\forall a, b \in I$: $a \cap D(x)_C = \emptyset \Leftrightarrow b \cap D(x)_C = \emptyset$. Therefore, $\forall a \in I\ \#inc((x,a), P(x)) = \#inc(I, P(x))$.
5. Immediate from 2 and 4.

The problem is to compute the value $\#inc(I, P(x))$ for each $I \in P(x))$. It is not possible to compute *independently* each $\#inc(I, P(x))$ in $O(1)$, but we will show that all $\#inc(I, P(x))$ can be computed in $O(|P(x)|)$. Therefore, since $|\mathcal{I}(B(P(x)))| \leq 2*|P(x)|+1$, the amortized complexity for an $\#inc(I, P(x))$ is $O(1)$. The following property helps us to compute all $\#inc(I, P(x))$ in $O(|P(x)|)$:

Property 6 $\#inc(I, B(P(x))) = |P(x)| - |\{C / D(x)_C \cap I = I\}|$

Proof: $\#inc(I, B(P(x))) = |\{C \text{ s.t. } D(x)_C \cap I = \emptyset\}|$ and by Property 5 (point 3) $|\{C\ / D(x)_C \cap I = \emptyset\}| + |\{C / D(x)_C \cap I = I\}| = |P(x)|$.

The ranges $\mathcal{I}(B(P(x))$ are traversed w.r.t. the ascending order, applying for each $I_K = [a, b)$ the following rules to determine how many constraints are satisfied:

- if a is a minimum of a domain $D(x)_C$ then we enter in a new $D(x)_C$ and the range I_k satisfies one constraint more than the previous one, I_{k-1}.
- if $a-1$ is a maximum of a domain $D(x)_C$: we close a domain $D(x)_C$ and I_k satisfies one constraint less than I_{k-1}.

This idea has to be refined, given that a value in $D(x)$ may correspond to minima or maxima of several $D(x)_{C_i}$. Therefore we count the number of minima of domains equal to a, and maxima of domains equal to $a-1$.

Notation 3 *Given* $a \in D(x)$*.*

- $nMin(a, B(P(x)))$ *denotes the number of domains* $D(x)_C \in B(P(x))$ *such that* $min(D(x)_C) = a$*,*
- $nMax(a, B(P(x)))$ *denotes the number of domains* $D(x)_C \in B(P(x))$ *such that* $max(D(x)_C) = a$*.*

Property 7 *Given $I_k = [p, q)$. The number of satisfied constraints $\#sat(I_k, P(x))$ of a range $I_k \in \mathcal{I}(B(P(x)))$ can be computed recursively by:*

$$\#sat(I_k, P(x)) = \#sat(I_{k-1}, P(x)) + nMin(p, B(P(x))) - nMax(p-1, B(P(x)))$$

with $\#sat(I_{-1}, P(x)) = 0$.

Proof: By induction. The lower bound of I_0 is $min(D(x))$, because all $D(x)_C \subseteq D(x)$. Thus, $\#sat(I_0, P(x)) = \#sat(I_{-1}, P(x)) + nMin(min(D(x)), B(P(x))) - nMax(min(D(x)) - 1, B(P(x))) = nMin(min(D(x)), B(P(x)))$. Therefore the property holds for I_0. Suppose that it is true for I_{k-1}, and $I_k = [p, q)$. By definition of $nMin$ and $nMax$ and from Property 5 (point 3) we have $\#sat(I_k, P(x)) = \#sat(I_{k-1}, P(x)) + nMin(p, B(P(x))) - nMax(p-1, B(P(x)))$

From this proposition and Property 6 the following property holds:

Property 8 *Given $I = [p, q)$. Then, $\#inc(I, P(x)) = |P(x)| - \#sat(I, P(x))$.*

Consider again the previous example in order to illustrate this principle:

Interval I_k	$\#sat(I_k, P(x))$	$\#inc(I_k, P(x))$
$I_1 = [0, 3)$	0 + 1 - 0 = 1	2
$I_2 = [3, 6)$	1 + 0 - 1 = 0	3
$I_3 = [6, 8)$	0 + 1 - 0 = 1	2
$I_4 = [8, 11)$	1 + 1 - 0 = 2	1

We deduce $\#inc(x, P(x)) = min_{I \in P(x)}(\#inc(I_k, P(x)) = 1$. The following algorithm is a possible implementation of the computation of $\#inc$ counters:

```
computeMinInc(x, P(x), B(P(x)), I(B(P(x))))
    #sat ← 0;
    #inc(x, P(x)) ← |P(x)|;
    k ← 1;
    while k ≠ |I(B(P(x)))| do
            let I_k = [p, q) be the k^th range of I(B(P(x)));
            #sat ← #sat + nMin(p, B(P(x))) − nMax(p − 1, B(P(x)));
            if |P(x) − #sat| < #inc(x, P(x)) then
                #inc(x, P(x)) ← |P(x) − #sat|;
                k ← k + 1;
    return #inc(x, P(x));
```

4.3 Necessary condition of Consistency

The $LB_{\mathcal{P}(\mathcal{C})}$ is computed by performing a sum of all $\#inc(x, P(x))$, following a schema similar to PFC-MRDAC. Note that the LB provided by PFC-RDAC and RMA are the same when filtering algorithms reduce only the bounds of the domains.

Compared with PFC-MRDAC, the computation of the lower bound $LB_{\mathcal{P}(\mathcal{C})}$ of Property 2 remains the same. Thus, we have: $LB_{\mathcal{P}(\mathcal{C})} = \sum_{x_i \in var(\mathcal{C})} \#inc(x_i, P(x_i))$. The difference is the computation of all $\#inc(x_i, P(x_i))$, done trough the ComputeMinInc presented in above.

Complexity In order to compare the complexity of checking the consistency of a ssc with RMA and with PFC-RDAC (the non incremental version of PFC-MRDAC [2]), we insist on the fact that the principle is the same except for computing the $\#inc(x, P(x))$. Consequently, we compare the procedure COMPUTEMININC with the method used in PFC-RDAC for computing $\#inc(x, P(x))$:

Notation 4 *We denote by :*

- *d the maximal domain size: $max_{x\in\mathcal{X}}(|D(x)|)$,*
- *p the maximal size of a set in the var-partition: $max_{P(x)\in\mathcal{P}(\mathcal{C})}(|P(x)|)$,*
- *f the maximal complexity of a filtering algorithm of a constraint C applied on a domain $D(x)$ (useful either to know if a value $a \in D(x)$ is consistent with a constraint C, in this case we call the algorithm with $D(x) = \{a\}$, or to compute $D(x)_C$).*

In PFC-RDAC, a counter $\#inc((x, P(x), a))$ of violations involved by each value a of $D(x)$ is computed. Thus, for each $a \in D(x)$ the complexity is $O(f * |P(x)|)$; and then we iterate over all counters to determine the minimal one, $\#inc(x, P(x))$. Finally the complexity is $O(|D(x)| * f * |P(x)|)$.

In RMA, the filtering algorithm of each constraint is called only once. That is, $|P(x)| * f$. We sort the $|P(x)| * 2 + 2$ bounds to compute $\mathcal{I}(B(P(x)))$: $O(|P(x)| * log(|P(x)|))$; and we iterate on them to determine $\#inc(x, P(x))$. Finally the complexity is $O(|P(x)| * log(|P(x)|) + f * |P(x)|)$.

To obtain LB, these computations will be performed at most n times (n is the number of variables). Since $\sum_{x_i\in X(\mathcal{C})}(|P(x_i)|) = |\mathcal{C}|$ we have:

	PFC-RDAC	RMA
$\#inc(x, P(x))$	$O(\|D(x)\| * f * \|P(x)\|)$	$O(\|P(x)\| * log(\|P(x)\|) + f * \|P(x)\|)$
$LB_{\mathcal{P}(\mathcal{C})}$	$O(\|\mathcal{C}\| * d * f)$	$O(\|\mathcal{C}\| * (log(p) + f))$
$LB_{\mathcal{P}(\mathcal{C})}$ when f is negligible	$O(\|\mathcal{C}\| * d)$	$O(\|\mathcal{C}\| * log(p))$

Note that $p \leq |\mathcal{C}|$ (if the partition is uniform we have even $p = |\mathcal{C}|/n$). RMA is interesting when the number of values in domains is important w.r.t. the number of constraints. In other terms, in some cases using PFC-MRDAC is almost forbidden because domain sizes are too big, and then the RMA is a good alternative.

4.4 Filtering Algorithm

In this section we propose to apply the principle described in previous sections to filter domains.

It is important to note that if only range based filtering algorithms are associated with the constraints then it is useless to introduce a filtering algorithm which is able to create "holes" in the domains, because these "holes" will not be used by the other algorithms. However, this can have an interest for the search heuristic, which can be based on the number of remaining values in the domains of the variables.

We first propose an algorithm which reduce only the bounds, even if it is also possible to provide an algorithm which creates holes.

Following the schema of PFC-MRDAC (see Theorem 2), we will take into account the lower bound corresponding to all sets in the var-partition $\mathcal{P}(\mathcal{C})$ except $P(x)$: $LB_{\mathcal{P}(\mathcal{C}-P(x))}$. As in PFC-MRDAC, viability of values will be evaluated according to $max(D(obj))$:

Theorem 3 *Let $I \in \mathcal{I}(B(P(x)))$. If $\#inc(I, P(x)) + LB_{\mathcal{P}(\mathcal{C}-P(x))} > max\ (D\ (obj))$ then all values in I can be removed from $D(x)$.*

Proof: I is homogeneously inconsistent thus any value in I violated the same number of constraints. From theorem 2 the proof holds.

The algorithm is based on this theorem: starting from $I = [min(D(x)), q)$, we remove successive ranges while the condition of Theorem 3 is satisfied; and similarly, from the maximum of $D(x)$: starting from $I = [p, max(D(x)) + 1)$, we remove successive ranges while the condition of Theorem 3 is satisfied.

This algorithm can be slightly modified in order to obtain further domain reductions, that is, to create "holes" in the domain: for each range, Theorem 3 is applied and the domain is accordingly modified.

These algorithms have the same worst case complexity, and the gain on PFC-MRDAC is similar as the gain obtained for the consistency checking.

Once again, if the filtering algorithms associated with soft constraints are range based algorithms, the filtering algorithm which makes holes and PFC-MRDAC will lead to the same domain reduction.

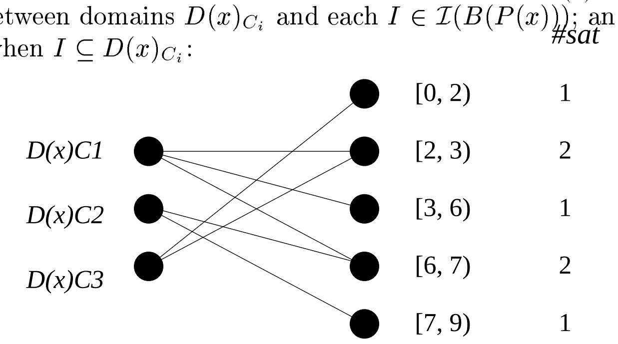
4.5 Incremental Version

Until now, we implicitly considered that all the computations are done from scratch. In this section, we study the incremental behavior of RMA. We suggest a technique for maintaining incrementally $\mathcal{I}(B(P(x)))$ during the search.

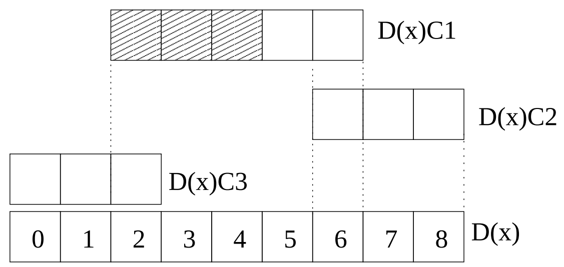
By propagation, the set of homogeneous ranges $\mathcal{I}(B(P(x)))$ can be modified and consequently $\#inc(x, P(x))$ can also be modified. We distinguish two cases: a modification of $D(x)$ and a modification of $D(x)_C$, $C \in P(x)$.

***$D(x)$* is modified** Assume the minimum of $D(x)$ is increased. We denote by $oldmin$ the previous value and $newmin$ the new one. To update $\mathcal{I}(B(P(x)))$ it is only required to remove ranges $[p, q) \in \mathcal{I}(B(P(x)))$ such that $q \leq newmin$ and, if $\exists\ [p, q)$ such that $p < newmin$ and $q > newmin$, increase the minimum of such range in order to transform it in a new range $[newmin, q)$.

A range $D(x)_C$ is modified For instance, let x be a variable such that $D(x) = [0, 8]$ and the constraints C_1, C_2, C_3 such that $D(x)_{C_1} = [2, 6]$, $D(x)_{C_2} = [6, 8]$, $D(x)_{C_3} = [0, 3]$. To illustrate the problem we use a bipartite graph $G_{P(x)}$, which represents the link between domains $D(x)_{C_i}$ and each $I \in \mathcal{I}(B(P(x)))$; an edge joins $D(x)_{C_i}$ and I when $I \subseteq D(x)_{C_i}$:



Assume that, due to the propagation process, $D(x)_{C_1}$ is modified to $[5, 6]$ (removed values from $D(x)_{C_1}$ are dashed in the following figure):



We need to update $G_{P(x)}$ according to this modification. We can explain our algorithm by describing its behavior on this example: for the first range previously connected to C_1 (vertex $D(x)_{C_1}$), that is $[2, 3)$, we remove the edge $(D(x)_{C_1}, [2, 3))$, and we decrease the counter $\#sat$ equal to the degree of vertex $D(x)_{C_1}$. The algorithm checks then if the vertex corresponding to the range which precedes $[2, 3)$, that is $[0, 2)$ is connected to the same vertices than $[2, 3)$ in $G_{P(x)}$. The answer is yes, so we perform the union of the two ranges, by grouping the two nodes into a single one: $[0, 3)$.

The process is repeated on next ranges until value 5 is reached. A new range $[5, 6)$ is then created, and the current range $[3, 6)$ is reduced to $[3, 5)$. The $\#sat$ counters are updated (one constraint less than $[3, 6)$ for $[3, 5)$, the same number for $[5, 6)$). $\mathcal{I}(B(P(x)))$ has been updated.

More formally, the following algorithm updates $\mathcal{I}(B(P(x)))$ after a modification of the minimum of $D(x)_C$. In practice, to deal efficiently with unions of ranges we maintain for each $I \in \mathcal{I}(B(P(x)))$ the set $\delta(I)$ of constraints in $P(x)$ that are satisfied exactly by one range among I and the range which precedes I in $\mathcal{I}$ $(B(P(x)))$. Implicitly if $\delta(I) = \emptyset$, $\#sat$, values of the two ranges are equal.

UPDATEMODIFMIN $(\mathcal{I}(B(P(x))),\ D(x)_C = [a,b),\ D'(x)_C = [newmin, b))$
    $I \leftarrow [p,q) \in \mathcal{I}(B(P(x)))$ such that $p = a$;
    while $q \leq newmin$ **do**
        $\#sat(I, P(x)) \leftarrow \#sat(I, P(x)) - 1$;
        if $\delta(I) = \emptyset$ **then**
            $I \leftarrow I \cup$ predecessor of I in $\mathcal{I}(B(P(x)))$;
        $I \leftarrow$ successor $[p,q)$ of I in $\mathcal{I}(B(P(x)))$;
    if $q < b$ **then**
        $I \leftarrow [p, newmin)$;
        add $[newmin, q)$ in $\mathcal{I}(B(P(x)))$;
        $\#sat([newmin, q), P(x)) \leftarrow \#sat(I, P(x))$;
    $\#sat(I, P(x)) \leftarrow \#sat(I, P(x)) - 1$;
    return $\mathcal{I}(B(P(x)))$;

In PFC-MRDAC an update of the number $\#inc((x,a), P(x))$ of the constraints violated by a value corresponds to one operation: $O(1)$. For k values the worst case complexity is of the order of $|D(x)|$ (when $k \simeq D(x)$). In the algorithm above if a domain $D(x)_C$ is reduced then the worst-case complexity for updating $\mathcal{I}(B(P(x)))$ is bounded by the number of ranges in $\mathcal{I}(B(P(x)))$. Therefore, the computation is faster than PFC-MRDAC when the reduction concerns a number of values greater that the number of ranges in $\mathcal{I}(B(P(x)))$. This number is less than or equal to $|D(x)|$.

5 Discussion

The technique we propose can be used when:

1. domains are not ranges
2. filtering algorithms associated with constraints are not range-based algorithms

Domains are not ranges We can either consider that domains are ranges even if they are not (RMA is then still valid but no longer equivalent to PFC-MRDAC), or directly deal with such domains. When we deal directly with such domains, each domain is an union of ranges. RMA can be applied, but the complexity of this algorithm changes because each range of a domain has to be taken into account. Thus, the number of ranges in $\mathcal{I}(B(P(x)))$ is accordingly changed and also the complexity of the algorithm. We note that the maximal number of ranges in of $\mathcal{I}(B(P(x)))$ is bounded by $min(|P(x)|, |D(x)|)$ and consequently by $|D(x)|$. In PFC-MRDAC, we maintain exactly $|D(x)|$ counters but they have not to be sorted, which is the case in RMA. The choice will then depend on the average number of ranges compared with $|D(x)|$.

Filtering algorithms are not range-based algorithms The application of a filtering algorithm for a constraint may lead to domains which are not ranges. That is, a domain correspond to several ranges. The two possibilities that can be studied are the same of the previous case: either the “holes” are ignored or the algorithm deals with the set of ranges corresponding to each domain.

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7 Conclusion

We propose an alternative to PFC-MRDAC called the Range-based Max-CSP Algorithm (RMA), based on the exploitation of bound-based filtering algorithms of constraints. An incremental version of the algorithm is provided. This technique does not require to study each value of each domain: its complexity depends only on the number of variables and constraints. It is more suited to problems with large domains than PFC-MRDAC, which requires too much memory. When the only information available for a variable x w.r.t. a constraint C is the values of bounds of $D(x)$ when the filtering algorithm of C is applied, the lower bounds provided by PFC-MRDAC and RMA are identical. In this case, RMA is faster than PFC-MRDAC when domains are large. Some variations of RMA are also studied in order to deal with more general cases, as domains with holes.

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