1.3. Simplification and extension of the SPREAD Constraint

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Many constraint satisfaction problems like the Balanced Academic Curriculum Problem (BACP, problem 30 of CSPLib) require the solution to be balanced. The goal of BACP is to assign periods to courses such that the academic load of each period is balanced, i.e., as similar as possible. The most balanced solution will depend on the chosen criterion. Given a set of variables $X = \{x_1, ..., x_n\}$, a first criterion used in [BRA 02] to solve BACP is to minimize the largest deviation from the mean: $\max_{x \in X} |x - \mu|$. An alternative one is to minimize the sum of square deviations from the mean: $\sum_{x \in X} (x - \mu)^2$. The minimization of one criterion does not imply the minimization of the second one. Nevertheless, the sum of square deviations probably corresponds better to the intuitive notion of balance measure and is commonly used in statistics. SPREAD recently introduced by Pesant and Régin [GIL 05] constraints the mean and the sum of square deviations of a set of variables. The particular case of SPREAD with a fixed mean is considered here. Given a set of finite-domain (discrete) variables $X = \{x_1, ..., x_n\}$, one value μ and one interval variable π, $\texttt{SPREAD}(X, \mu, \pi)$ holds if $n.\mu = \sum_{i=1}^{n} x_i$ and $\pi = \sum_{i=1}^{n} (x_i - \mu)^2$. For the constraint to be consistent, $n.\mu$ must be an integer. As a consequence $n^2.\pi$ is also an integer. Pesant and Régin [GIL 05] propose a filtering algorithm of π from X and μ, and of X from $\pi^{\max}$. We extend these results and describe a simpler filtering algorithm on X with the same $\mathcal{O}(n^2)$ complexity achieving bound-consistency with respect to $\pi^{\max}$ and μ.

Section 1.3.1 recall the filtering of π [GIL 05] and Section 1.3.2 describes the filtering on X.

1.3.1. *Filtering of* π

Let define S_μ the set of tuples x satisfying the following constraints:

$$\sum_{i=1}^{n} x_i = n.\mu \qquad [1.1]$$

$$x_i^{\min} \leq x_i \leq x_i^{\max}, \quad \forall i \in [1, .., n] \qquad [1.2]$$

The filtering of π is based on the optimal values $\underline{\pi} = \min\{\sum_{i=1}^{n} (x_i - \mu)^2 \ : x \in S_\mu\}$ and $\overline{\pi} = \max\{\sum_{i=1}^{n} (x_i - \mu)^2 \ : x \in S_\mu\}$. Computing $\overline{\pi}$ can be shown to be $\mathcal{NP}$-hard. Instead of the exact value, an upper bound $\overline{\pi}^{\uparrow}$ can be used to make filtering. An upper bound obtained from the relaxed problem without the mean constraint [1.1] is $\overline{\pi}^{\uparrow} = \sum_{i=1}^{n} \left(\max\left(|x_i^{\max} - \mu|, |x_i^{\min} - \mu|\right)\right)^2$. The filtering on the domain of π is $Dom(\pi) \longleftarrow Dom(\pi) \cap [\underline{\pi}, \overline{\pi}^{\uparrow}]$.

The rest of this section describes the algorithmic solution from Pesant and Régin [GIL 05] to compute $\underline{\pi}$.

An optimal tuple in the problem of finding $\underline{\pi}$ has the property to be a v-centered assignment on each variable: $x_i := x_i^{\max}$ if $x_i^{\max} \leq v$, $x_i := x_i^{\min}$ if $x_i^{\min} \geq v$ and $x_i := v$ otherwise. The optimization problem is now reduced to finding a value v such that the v-centered assignment on every variable $x \in X$ respects the constraint [1.1]. The value v can be anywhere in $[\min_X x^{\min}, \max_X x^{\max}]$. A splitting of this interval into a set of $\mathcal{O}(n)$ contiguous intervals $\mathcal{I}(X) = \{[I_1^{\min}, I_1^{\max}], [I_2^{\min}, I_2^{\max}], ...\}$ permits us to find v by iterating once over this set. The construction of this set is described below.

Let $B(X)$ be the sorted sequence of bounds of the variables of X, in non-decreasing order and with duplicates removed. Define $\mathcal{I}(X)$ as the set of intervals defined by a pair of two consecutive elements of $B(X)$. The k^{th} interval of $\mathcal{I}(X)$ is denoted by I_k. For an interval I_k, $prev(I_k) = I_{k-1}$ $(k > 1)$ and $succ(I_k) = I_{k+1}$. For example, let $X = \{x_1, x_2, x_3\}$ with $x_1 \in [1, 3]$, $x_2 \in [2, 6]$ and $x_3 \in [3, 9]$ then $\mathcal{I}(X) = \{I_1, I_2, I_3, I_4\}$ with $I_1 = [1, 2]$, $I_2 = [2, 3]$, $I_3 = [3, 6]$, $I_4 = [6, 9]$ and $prev(I_3) = I_2$, $succ(I_3) = I_4$.

Let assume that the value v of the optimal solution lies in the interval $I \in \mathcal{I}(X)$. Sets $R(I)$ and $L(I)$ are defined as $R(I) = \{x | x^{\min} \geq \max(I)\}$ and $L(I) = \{x | x^{\max} \leq \min(I)\}$. The optimal solution is a v-centered assignment, hence all variables $x \in L(I)$ take their value $x^{\max}$ and all variables in $R(I)$ their value $x^{\min}$. It remains to assign the variables subsuming I denoted by $M(I) = \{x | I \subseteq I_D(x)\}$ and the cardinality of this set by $m = |M(I)|$. In a v-centered assignment with $v \in I$, the variables in $M(I)$ must take a common value v. The sum constraint [1.1] can be reformulated with the introduced notations as $\sum_{x \in R(I)} x^{\min} + \sum_{x \in L(I)} x^{\max} + \sum_{x \in M(I)} v = n.\mu$ or more simply as $v^* = (n.\mu - ES(I))/m$ where $ES(I) = \sum_{x \in R(I)} x^{\min} + \sum_{x \in L(I)} x^{\max}$. The value v^* is admissible only if $v^* \in I$. This constraint is satisfied if $n.\mu \in V(I) = [ES(I) + \min(I).m, ES(I) + \max(I).m]$. For two consecutive intervals $I_k, I_{k+1} \in \mathcal{I}(X)$, intervals $V(I_k)$ and $V(I_{k+1})$ are also contiguous: $\min(V(I_{k+1})) = \max(V(I_k))$. As a consequence, for every consistent value μ, there exists one interval $I \in \mathcal{I}(X)$ such that $n.\mu \in V(I)$. The procedure to find $\underline{\pi}$ can be easily described and can be computed in linear time given $\mathcal{I}(X)$ and the x_i's sorted according to their bounds [GIL 05].

1) Find $I \in \mathcal{I}(X)$ such that $n.\mu \in V(I)$. This interval is denoted I^μ.

2) Compute $v = (n.\mu - ES(I^\mu))/m$.

3) The optimal solution is the v-centered assignment uniquely defined by v.

1.3.2. *Filtering of* X

Let define $S_{\mu\pi}$ the set of tuples x satisfying the following constraints:

$$\sum_{i=1}^{n} x_i = n.\mu$$
$$\sum_{j=1}^{n} (x_j - \mu)^2 \leq \pi^{\max}$$
$$x_i^{\min} \leq x_i \leq x_i^{\max}, \quad \forall i \in [1,..,n]$$

The filtering on X is based on the optimal values $\overline{x}_i = \max\{x[i] \ : \ x \in S_{\mu\pi}\}$ and $\underline{x}_i = \min\{x[i] \ : \ x \in S_{\mu\pi}\}$. Finding $\overline{x}_i$ and $\underline{x}_i$ are symmetrical problems with respect to μ, hence only the former is considered here. The optimal value $\overline{x}_i$ can be found by shifting all the domain of x_i until the minimization of the sum of square deviations gives $\underline{\pi} = \pi^{\max}$. More formally, the maximization problem can be transformed into an equivalent problem by renaming $x_i = x_i^{\min} + d_i$. The objective of this equivalent problem is $\overline{d}_i = \max(d_i)$ with $0 \leq d_i \leq x_i^{\max} - x_i^{\min}$. The algorithm shown on Figure 1.1 computes $\overline{d}_i$ in $\mathcal{O}(n)$.

The procedure starts from the optimal value $\underline{\pi}$. The problem of finding $\underline{\pi}$ is then modified by increasing all values from the domain of variable x_i by a non negative value d_i. Let denote the variable with modified domain by x_i', the modified set of variables by X' and the corresponding quantities by $ES'(I^{\mu})$ and $V'(I^{\mu})$. For a variable $x_i \in R(I^{\mu}) \cup M(I^{\mu})$, the new optimal value $\underline{\pi}'$ increases quadratically with d_i. The procedure of modifying the domain of x_i is repeated at most $\mathcal{O}(n)$ times until $\underline{\pi}' = \pi^{\max}$.

Assume first that $x_i \in R(I^{\mu})$. After a shift of d_i, $ES'(I^{\mu}) = ES(I^{\mu}) + d_i$ and $V'(I^{\mu}) = V(I^{\mu}) + d_i$. If $d_i \leq \Delta = n.\mu - \min(V(I^{\mu}))$, the value v' of the v-centered assignment remains in I^{μ} but becomes $v' = v - d_i/m$. Consequently, the new optimal value becomes $\underline{\pi}' = \left(\sum_{x_j \in L(I^{\mu})} (x_j^{\max})^2\right) + \left(\sum_{x_j \in R(I^{\mu})} (x_j^{\min})^2\right) + d_i^2 + 2.d_i.x_i^{\min} + \left(\sum_{x_j \in M(I^{\mu})} (v - \frac{d_j}{m})^2\right) - n.\mu^2 = \underline{\pi} + d_i^2 + 2.d_i.x_i^{\min} + m\left(\frac{d_i^2}{m^2} - 2\frac{d_i}{m}v\right)$. Recall that the problem is to find $\overline{d}_i$ such that $\underline{\pi}' = \pi^{\max}$. Hence, $\overline{d}_i^*$ is the positive solution of the second degree equation $a.d_i^2 + 2.b.d_i + c = 0$, where $a = (1 + \frac{1}{m})$, $b = x_i^{\min} - v$ and $c = \underline{\pi} - \pi^{\max}$. We have $\overline{d}_i = \overline{d}_i^*$ only if $\overline{d}_i^* \leq \Delta$ because otherwise the v-centered assignment does not remain in I^{μ}. If $\overline{d}_i^* > \Delta$ then x_i can be shifted by Δ so that the value of the v-centered assignment lies in $prev(I^{\mu})$, and repeat the procedure on the new problem. The process stops when I_1 is reached or if $\overline{d}_i^* \leq \Delta$.

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Algorithm: FindDMax(x_i, I^μ)
Data: x_i ∈ R(I^μ); I^μ ∈ 𝓘; n.μ ∈ V(I^μ);
Result: d̄_i such that π′ = π^max with x′ = x + d̄_i
if M(I^q) = φ then
 | return FindDMax(x_i, prev(I^μ))
end
Δ = n.μ − min(V(I^μ))
d̄*_i = (−b+√(b²−ac))/a          /* values a, b, c are defined in the text */
if d̄*_i ≤ Δ then
 | return d̄*_i
else
 | if I^μ = I_1 then
 | | return Δ
 | else
 | | return Δ+FindDMax(x_i + Δ, prev(I^μ))
 | end
end
```

Figure 1.1. *Algorithm to find $\overline{d}_i$ when $x_i \in R(I^\mu)$.*

Complete algorithm is shown on Figure 1.1 and runs in $\mathcal{O}(n)$ since there are at most $|\mathcal{I}(\mathcal{X})| < n$ recursive calls and that the body executes in constant time.

Finding $\overline{d}_i$ for $x_i \in M(I^\mu)$ reduces easily to the previous case. When x_i is increased by d_i, the optimal assignment does not change while $d_i \leq v - x_i^{\min}$. For $d_i = v - x_i^{\min}$ two new intervals are created replacing I^μ. These are $I_j = [\min(I^\mu), v]$ and $I_k = [v, \max(I^\mu)]$ with $n.\mu = \max(V'(I_j)) = \min(V'(I_k))$. For this new configuration, the optimal assignment is the same but now $n.\mu \in V'(I_j)$ and $x'_i \in R(I_j)$. Hence $\overline{d}_i = v - x_i^{\min} + FindDMax(x'_i, I_j)$ where $x'_i = x_i + v - x_i^{\min}$.

1.3.3. *Conclusion*

`SPREAD` is a balancing constraint for the criterion of sum of square deviations from the mean. Filtering algorithms associated with it have been proposed by [GIL 05]. We have shown that simpler filtering algorithms with the same efficiency can be designed when the mean is fixed. We currently work on an implementation of `SPREAD`.

1.4. Bibliography

[BRA 02] BRAHIM HNICH ZEYNEP KIZILTAN T. W., "Modelling a Balanced Academic Curriculum Problem", *Proceedings of CP-AI-OR-2002*, 2002.

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